

Application of Sequential Nonparametric Confidence Bands in Finance

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Abstract

In a nonparametric setting, the functional form of the relationship between the response variable and the associated predictor variables is assumed to be unknown when data is fitted to the model. Non-parametric regression models can be used for the same types of applications such as estimation, prediction, calibration, and optimization that traditional regression models are used for. The main aim of nonparametric regression is to highlight an important structure in the data without any assumptions about the shape of an underlying regression function. Hence the nonparametric approach allows the data to speak for itself. Applications of sequential procedures to a nonparametric regression model at a given point are considered. The primary goal of sequential analysis is to achieve a given accuracy by using the smallest possible sample sizes. These sequential procedures allow an experimenter to make decisions based on the smallest number of observations without compromising accuracy. In the nonparametric regression model with a random design based on independent and identically distributed pairs of observations (X, Y) , where the regression function $m(x)$ is given by $m(x) = E(Y | X = x)$, estimation of the Nadaraya-Watson kernel estimator $(m_{NW}(x))$ and local linear kernel estimator $(m_{LL}(x))$ for the curve $m(x)$ is considered. In order to obtain asymptotic confidence intervals for $m(x)$, two stage sequential procedure is used under which some asymptotic properties of Nadaraya-Watson and local linear estimators have been obtained. The proposed methodology is first tested with the help of simulated data from linear and nonlinear functions. Encouraged by the preliminary findings from simulation results, the proposed method is applied to estimate the nonparametric regression curve of CAPM.

1. Introduction

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be a sequence of iid bivariate random variables from a population with a distribution function $f(x, y)$ and for simplicity we assume that $X_i \in (0,1)$ with density function $f(x)$. Consider the nonparametric regression model

$$Y_i = m(x_i) + \epsilon_i, \quad i = 1, \dots, n \quad (1.1)$$

where ϵ_i is a sequence of iid random variables with $E(\epsilon_i) = 0$, $E[\epsilon_i^2] = \sigma^2$ and $m(x)$ is an unknown function. We identify two distinct types of design, equidistant design where $x_i = i/n$ for $(1 \leq i \leq n)$ and random design, where x_i 's arises from independent realizations of a distribution having density f on $[0,1]$. In this situation, we consider the random design regression model and call X the design variable and Y the response variable. Our main concern is the estimation of fixed width confidence bands for the unknown function $m(x)$ for a given x_0 . Estimators for $m(x)$ is based on a kernel estimator $\hat{m}(x)$ for the regression curve $m(x)$. We use Nadaraya-Watson estimator, $\hat{m}_{NW}(x)$ and local linear estimator, $\hat{m}_{LL}(x)$ which are respectively defined by

$$\hat{m}_{NW}(x) = \frac{\sum_{i=1}^n Y_i K\left(\frac{x - X_i}{h_n}\right)}{\sum_{i=1}^n K\left(\frac{x - X_i}{h_n}\right)} \quad (1.2)$$

and

$$\hat{m}_{LL}(x) = \frac{\sum_{i=1}^n w_i Y_i}{\sum_{i=1}^n w_i} \quad (1.3)$$

Here $K(\cdot)$ is a kernel function, h_n a bandwidth or smoothing parameter and

$$w_i = K\left(\frac{x - x_i}{h_n}\right) (s_{n,2} - (x - x_i)s_{n,1}) \quad (1.4)$$

with

$$s_{n,l} = \sum_{i=1}^n K \left(\frac{x - x_i}{h_n} \right) (x - x_i)^l, \quad l = 1, 2 \quad (1.5)$$

In general local polynomial smoother (Fan and Gijbels, 1996) being superior in some respects (Fan, 1993), recent contributions by Boularan et al. (1995), Liebscher (1998), Einmahl and Mason (2000) as well as Qian and Mammitsch (2000), among others, have given evidence of continuing interest in the Nadaraya-Watson estimator. One of the strengths of this estimator certainly consists in its automatic adaptation to designs where the local polynomial estimator may turn unreliably since its variance can fail to exist in random designs. Also, the Nadaraya-Watson estimator retains some optimality properties as exposed in Hardle and Marron (1985).

Methods for obtaining confidence bands for $m(x)$ can be found in Hall and Titterton (1988), Hardle (1989), Eubank and Speckman (1993) and Diebolt(1995). The most widely used confidence band for $m(x)$ is based on the theorem of Bickel and Rosenblatt (1973) for kernel estimation of a density function. Bias-corrected confidence bands for general nonparametric regression models are considered by Xia (1998).

In principle, confidence intervals can be obtained from asymptotic normality results for $\hat{m}(x)$. However, the limiting bias and variance depend on unknown quantities which have to be estimated consistently in order to construct asymptotic confidence intervals.

The capital asset pricing model (CAPM) implies that the expected return of an asset must be linearly related to the covariance of its return with the return of the market portfolio. β is defined as the gradient of the least squares linear regression where the excess return on the market over the risk-free rate is the predictor and the excess return on the asset over the risk-free rate is the response. This article explores a statistical analysis of historical data and develops a robust sequential nonparametric estimation of the CAPM that can be used when

the underlying assumptions (Campbell, 1997) fail. The possibility of existing a nonlinear relationship between the excess returns of an asset and a market is justified in the discussions in Long (1990), Luenberger (1993, 1998) and Efromovich S. (2004). Generally a fixed sample size is used to calculate the β 's. The sample size may be too large for some periods of time and too small for others. The larger the period is the more outdated is the information that is being used. The current situation in the Australian market is different from what it was in the early 1990's. The period of the statistical analysis should be as short as possible to minimize the effect of factors such as size of the institution, dividend per share, business environment, union actions etc. This is why sequential analysis appears to be suitable as its primary goal is to achieve a given accuracy by using the smallest possible sample size. A data-driven sequential nonparametric procedure which allows the investor to analyse the relationship between the excess rate of returns on an asset and the excess rate of returns on the market using the shortest period of historical data is proposed.

2. Framework

Throughout the present work, we will consider the following regression model with a random design. Let (X_i, Y_i) be an i.i.d. sample of observations taken a bivariate random vector (X, Y) with unknown bivariate distribution such that Y is integrable and

$$m(x) = E(Y|X = x), \quad x \in R \quad (2.1)$$

be the unknown regression function which describes the dependence of the so-called response variable Y on the value of X .

The following assumptions are used in this study. For more details we refer to Wand and Jones (1995):

- (i) $m''(x)$ is continuous for all $x \in [0,1]$.
- (ii) $K(x)$ is symmetric about $x = 0$ and supported on $[-1,1]$.

(iii) $h_n \rightarrow 0$ and $nh_n \rightarrow \infty$ as $n \rightarrow \infty$.

(iv) The given point $x = x_0$ must satisfy $h_n < x_0 < 1 - h_n$ for all $n \geq n_0$ where n_0 is fixed number.

The obvious problem that occurs when using such estimators is the choice of bandwidth, h_n . A representation of the bandwidth is derived with respect to the assumption that the point x_0 which the estimation is taking place satisfies $h_n < x_0 < 1 - h_n$ for all $n \geq n_0$ where n_0 is fixed. As in Isogai (1987a) and Mukhopadhyay (1999), we take $h_n = n^{-r}$ for $0.2 < r < 1$. Now using the property $h_n < x_0 < 1 - h_n$ one can prove that

$$r_0 = \max \left\{ 0.21, \frac{-\log[\min(x_0, 1 - x_0)]}{\log(n)} \right\} \text{ and } n \geq 4. \quad (2.2)$$

Let us make the standard assumptions regarding the kernel $K(\cdot)$ as in Silverman (1986), namely that

(i) $K(\cdot)$ is a bounded probability density function on the real line

(ii) $\lim_{|t| \rightarrow \infty} |t|K(t)$

(iii) $\int_{-\infty}^{+\infty} tK(t)dt = 0$

(iv) $\int_{-\infty}^{+\infty} t^2K(t)dt = k_2 < \infty$

A natural way of constructing a confidence band for $m(x)$ is follows. Suppose that $\hat{m}(x)$ is an estimator of $m(x)$ then $100(1 - \alpha)\%$ confidence band is of the form

$$\lim_{n \rightarrow \infty} \left[\Pr \left\{ |\hat{m}(x) - m(x)| \leq d, \forall x \in [0,1] \right\} \right] \geq 1 - \alpha \quad (2.3)$$

There are many difficulties with finding a good solution to inequality 2.3. Firstly, we must derive the asymptotic distribution of $\hat{m}(x) - m(x)$; secondly the estimation of residual variance and distribution function of X. Consequently, a good estimator of bandwidth is needed.

Theorem 1

Let us choose kernel $K(\cdot)$ such that $\int |K(t)| dt \leq \infty$, $\lim_{|t| \rightarrow \infty} tK(t) = 0$ and $\int |K(t)|^{2+\eta} dt \leq \infty$, for some $\eta > 0$ and the bandwidth h_n satisfies $\overline{\lim} nh_n < \infty$. Suppose $m(x)$ is twice continuously differentiable at $x = x_0$ and $E[|Y|^{2+\eta} | x = x_0]$ exists, then

$$\sqrt{nh_n} [\hat{m}(x) - m(x) - Bias] \simeq N\left(0, \frac{B_n \sigma^2}{f(x)}\right) \quad (2.4)$$

In general the bias of the LL estimator is smaller than NW estimator. However one can choose h_n such that $Bias \rightarrow 0$ as $n \rightarrow \infty$ for both these estimators and hence

$$\lim_{n \rightarrow \infty} \sqrt{nh_n} [\hat{m}(x) - m(x)] \simeq N\left(0, \frac{B_n \sigma^2}{f(x)}\right) \quad (2.5)$$

Note that $B_n = \int_{-\infty}^{+\infty} K^2(t) dt$. Suppose that we employ the standard normal kernel $K(t) = \{\sqrt{2\pi}\}^{-1} \exp(-t^2/2)$ with $-\infty < t < \infty$. This selected kernel function satisfies the above mentioned standing assumptions and $B_n = \{2\sqrt{\pi}\}^{-1}$.

3. Two-Stage Procedure

Given $d(> 0)$, $\alpha \in (0,1)$, and $h_n = n^{-r}$ with $0.2 < r < 1$, suppose that we wish to claim

$$P\{ m(x_0) \in I_n = [\hat{m}(x_0) \pm d] \} \geq 1 - \alpha \quad (3.1)$$

here x_0 is a fixed value. Using Theorem 1 one can see that the probability requirement (3.1)

can be met if $n \geq n^*$ where

$$n^* = \left\{ \frac{Z_{n,\alpha/2}^2 B_n \sigma^2}{d^2 f(x)} \right\}^{\frac{1}{1-r}} \quad (3.2)$$

and $z_{\alpha/2}$ is the upper 50 α % of the standard normal distribution. Now from the optimal sample size, n^* given in (3.2), the stopping rule is

$$N = \max\left\{n_0, \left\langle \left\{ t_{n,\alpha/2}^2 B_n \hat{\sigma}^2 d^{-2} f(x)^{-1} \right\}^{\frac{1}{1-r_1}} \right\rangle + 1 \right\} \quad (3.3)$$

where $t_{\alpha/2,n}$ is the upper 50 α % of the t-distribution with n degrees of freedom and from (2.2),

$$r_1 = \max\left\{0.21, -\log(\min[x_0, 1-x_0]) / \log n_0\right\}.$$

In the initial stage, we start with observations $(x_1, y_1), \dots, (x_{n_0}, y_{n_0})$ of size n_0 and estimate required final sample size N . If $N = n_0$, then we need no more observations in the second stage. Otherwise, we take extra sample of size $N - n_0$ in the second stage.

In an application of the stopping rule (3.3), it is important to select the best available values for the design constants r and n_0 for fixed predesigned values of d and α .

4. Variance Estimator Based on Curve Fitting

Hall and Marron (1990) introduced a variance estimator based on the mean square of a sequence of residuals and applies equally to both fixed and random types of design. The estimator is valid for general error distributions, has elementary bias and variance properties and is optimal to both first and second orders.

Estimators $\hat{\sigma}^2$ based on ath-order differences have the property that

$$E[\hat{\sigma}^2 - \sigma^2] \sim n^{-1} c_a \text{Var}(\epsilon^2) \quad (4.1)$$

where $1 \leq c_a \leq \infty$.

It is shown that there exist simple estimators which are applicable whenever the error distribution has finite fourth moment, and which achieve $c = 1$ in (5.3). They have minimum mean squared error in both first and second order senses. $\hat{\sigma}^2$ is simply the mean square of a

sequence of residuals. Where the mean function m has a derivatives, $\hat{\sigma}^2$ is constructed so that as n increases, $E[\hat{\sigma}^2 - \sigma^2] = n^{-1} \{ \text{Var}(\epsilon^2) + o(n^{-(4a-1)/(4a+1)}) \}$.

Let $\delta_{ij} = K \{ (x_i - x_j)/h_n \} / \sum_{k=1}^n K \{ (x_i - x_k)/h_n \}$. Residuals are defined by

$$\hat{\epsilon}_i = Y_i - \sum_{j=1}^n \delta_{ij} Y_j \quad (1 \leq i \leq n) \quad (4.2)$$

and let $S = \sum_i \hat{\epsilon}_i^2$. When the mean function m is zero, $E[S] = v\sigma$ where

$v = n - 2 \sum_i \delta_{ii} + \sum \sum_{ij} \delta_{ij}^2$. Therefore we define

$$\sigma_{HM}^2 = v^{-1} S = \left(n - 2 \sum_i \delta_{ii} + \sum \sum_{ij} \delta_{ij}^2 \right)^{-1} \sum_{i=1}^n \left(Y_i - \sum_{j=1}^n \delta_{ij} Y_j \right)^2 \quad (4.3)$$

σ_{HM}^2 has optimal first and second order properties (Hall and Marron, 1990). It may be proved that if $h_n \rightarrow 0$ and $nh_n \rightarrow \infty$ the $\sigma_{HM}^2 - E[\sigma_{HM}^2]$ is asymptotically normally distributed with mean zero and variance $n^{-1} \text{Var}(\epsilon_i^2)$.

5. Capital Asset Pricing Model (CAPM)

The development of CAPM helps economists to quantify risk and reward for bearing risky investments. Markowitz's (1959) mean-variance portfolio theory laid the groundwork for the CAPM. Sharpe (1964) and Lintner (1965b) built on Markowitz's work to develop economy-wide implications. The usual CAPM equation is a direct implication of the mean-variance efficiency of market portfolio. The CAPM assumes the existence of lending and borrowing at a riskfree rate of interest. Under this assumption the CAPM we have for the expected return of asset i ,

$$E[R_i] = R_f + \beta_{im}(E[R_m] - R_f); \quad \text{where } \beta_{im} = \frac{\text{Cov}[R_i, R_m]}{\text{Var}[R_m]} \quad (5.1)$$

Where R_i is the return of asset, R_m is the return on the market portfolio and R_f is the return on the risk free asset. Let us examine now why the above equation is called a pricing model. Suppose that an asset is purchased at price P_0 and later sold at price P_1 . The rate of return is then $R = (P_1 - P_0) / P_0$. Here P_1 is random so the CAPM implies

$$P_0 = \frac{E[P_1]}{1 + R_f + \beta(R_m - R_f)} \quad (5.2)$$

The CAPM can be useful for applications requiring a measure of expected returns. Some applications include cost of capital estimation, portfolio performance evaluation, and even-study analysis.

6. Simulation Results

A simulation study was conducted to compare the 95% fixed-width confidence intervals using NW and LL estimators. Simulations were performed using the

- Model (I): $m(x) = \sqrt{4x+3} + \varepsilon_i$ with $\sigma^2 = 0.25$ and
- Model (II): $m(x) = 2 \exp\{-x^2/0.18\} + 3 \exp\{-(x-1)^2/0.98\} + \varepsilon_i$ with $\sigma^2 = 0.25$

Where $\varepsilon_i \stackrel{iid}{\sim} N(0, \sigma^2)$.

In both cases, 15000 simulation replications were carried out to obtain the final sample sizes required to estimate $m(x)$ at $x_0=0.306$ ($m(x_0) = 2.0552$ and 3.0240 for models I and II respectively) given fixed-width, $2d$. The initial sample size is fixed at $n_0 = 25$. The following tables give the summary results obtained from the simulation study. Here \bar{p} is the coverage probability, \bar{n} is the average final sample size and $(.)$ gives the standard error of the estimated value.

Table 1: Simulation Results for the Model (I).

| d | n^* | \bar{n} | $\hat{m}_{NW}(x_0)$ | \bar{P}_{NW} | $\hat{m}_{LL}(x_0)$ | \bar{P}_{LL} | $\hat{\sigma}^2_{n_0}$ |
|------|-------|-----------|---------------------|--------------------|---------------------|--------------------|------------------------|
| 0.15 | 52 | 70 | 2.0966 | 0.9217 (0.0022) | 2.0469 | 0.9390 (0.0020) | 0.2653 |
| 0.13 | 82 | 110 | 2.1080 | 0.9019 (0.0024) | 2.0462 | 0.9473 (0.0018) | 0.2649 |
| 0.11 | 139 | 182 | 2.1205 | 0.8441 (0.0030) | 2.0477 | 0.9653 (0.0015) | 0.2617 |
| 0.09 | 263 | 349 | 2.1185 | 0.7944 (0.0033) | 2.0479 | 0.9779 (0.0012) | 0.2644 |
| 0.07 | 584 | 777 | 2.0956 | 0.9019 (0.0024) | 2.0469 | 0.9894 (0.0008) | 0.2650 |
| 0.05 | 1698 | 2260 | 2.0763 | 0.9582 (0.0016) | 2.0480 | 0.9962 (0.0005) | 0.2649 |

Table 2: Simulation Results for the Model (II).

| d | n^* | \bar{n} | $\hat{m}_{NW}(x_0)$ | \bar{P}_{NW} | $\hat{m}_{LL}(x_0)$ | \bar{P}_{LL} | $\hat{\sigma}^2_{n_0}$ |
|------|-------|-------------------|---------------------|--------------------|---------------------|--------------------|------------------------|
| 0.15 | 52 | 66.93 (0.25) | 3.0002 | 0.9355 (0.0020) | 3.0220 | 0.9334 (0.0020) | 0.2577 |
| 0.13 | 82 | 105.06 (0.40) | 3.0108 | 0.9559 (0.0017) | 3.0375 | 0.9464 (0.0018) | 0.258 |
| 0.11 | 139 | 180.46 (0.68) | 3.0044 | 0.9667 (0.0015) | 3.0368 | 0.9591 (0.0016) | 0.2602 |
| 0.09 | 263 | 337.03 (1.27) | 2.9925 | 0.9537 (0.0017) | 3.0306 | 0.9731 (0.0013) | 0.2583 |
| 0.07 | 584 | 759.76 (2.91) | 3.0029 | 0.9761 (0.0012) | 3.0319 | 0.9891 (0.0008) | 0.2608 |
| 0.05 | 1698 | 2149.35 (8.25) | 3.0012 | 0.9540 (0.0017) | 3.0274 | 0.9943 (0.0006) | 0.2563 |

The above tables clearly show that NW estimator fails to achieve the required coverage probability in the model I for small to moderate sample sizes. Whereas in model II it

performed as well as LL estimator by achieving the required coverage probability. However, LL estimator performed well in both models.

7. Application

Suppose that we observe n_0 pairs $\{(R_{M1}, R_{A1}), \dots, (R_{Mn}, R_{An0})\}$ where R_{Mi} ($=r_{Mi}-r_{fi}$) is the excess rate of return from the market during the i th period and R_{Ai} is the excess rate of return from an asset. Then, the nonparametric regression model we consider here is:

$$R_{Ai} = m(R_{Mi}) + \varepsilon_i, \quad i = 1, \dots, n_0 \quad (7.1)$$

Where $m(\cdot)$ is called the nonparametric regression function and ε_i s are random with zero mean and constant variance. The main aim of nonparametric regression is to highlight an important structure in the data without any assumption about the shape of the regression function.

Whereas in parametric approach one can assume linear regression model $f(x) = \beta_0 + \beta_1 x$ regardless of the data on hand.

The methodology developed in the previous sections has been applied to estimate CAPM using the excess rate of monthly returns of the Microsoft stock (R_{Ai}) and the excess rate of monthly returns of the market (R_{Mi}) whose proxy is Standard and Poor's 500 index which is a capital-weighted portfolio of most of the United State's largest stocks. The 13-weeks Treasury bill serves as the proxy for the risk-free asset.

We begin the analysis of data with visualization of a scatter diagram of the original data. The figure 1.A exhibits a scatter diagram for the excess rate of monthly returns of the Microsoft stock versus the excess rate of monthly returns of the market during a 55-month period that ended on May 1, 2007. If the CAPM is correct, then a linear relationship between these two rates with zero y-intercept should be observed. If the model is incorrect or its assumptions are invalid then a more complicated relationship may be visible. Here we examine whether a classical parametric regression analysis helps in our understanding of this data set. A most

commonly used parametric regression model is $m(x) = \beta_0 + \beta_1 x$. The least squares regression line is shown next in Figure 1.B.

Fig 1.A: Scatter diagram for Microsoft-SP500 Monthly Data,n=55

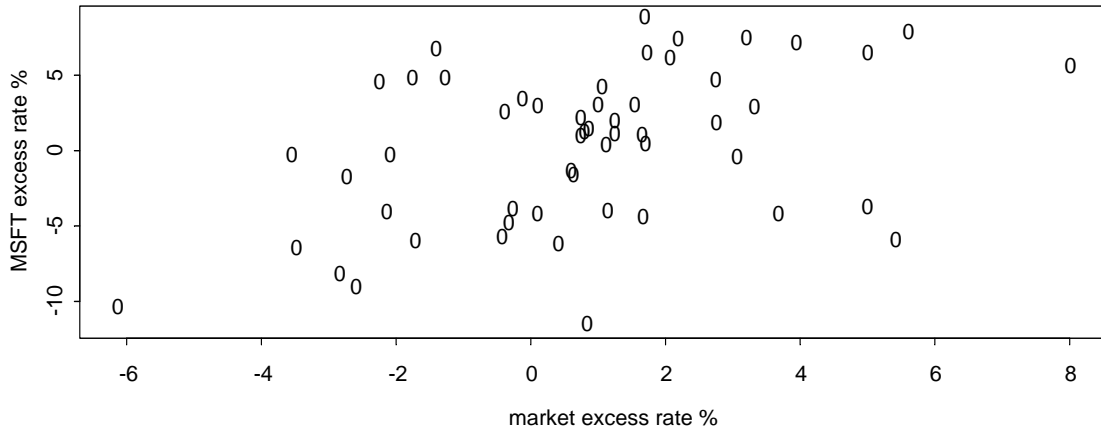


Fig 1.B: Linear Regression,beta=1.1672, y-intercept=-1.6110,n=55

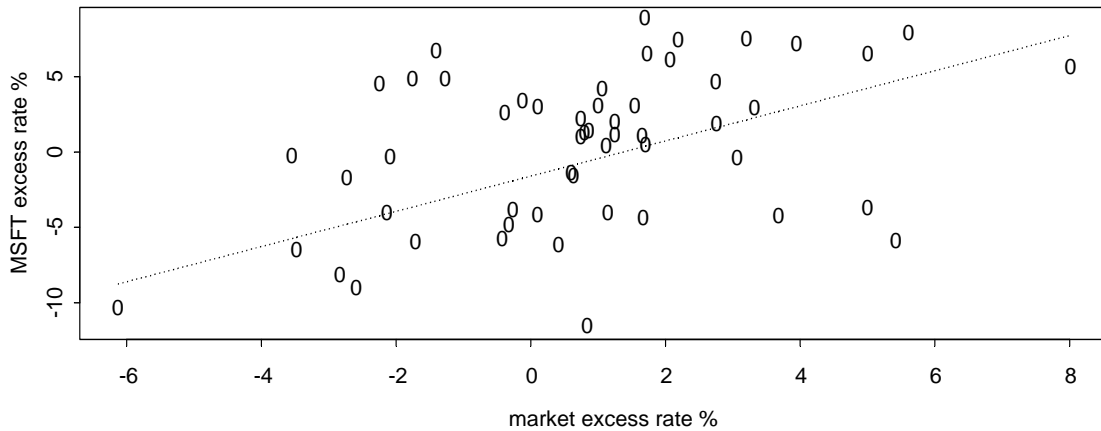


Figure 1.A&B: Linear regression based on the 55 months of Microsoft–SP500 monthly data.

The fitted line shows the fact that larger returns from the market imply larger returns from the stock and vice a versa. The parameters of the fitted regression model are as follows: the slope, $\hat{\beta}_1 = 1.4133$ and $\hat{\beta}_0 = -1.5845$. Standard error of $\hat{\beta}_0$, $SE(\hat{\beta}_0) = 1.13804$ and corresponding $P_{value} = 0.16954$ confirms non-zero y-intercept. This result contradicts CAPM.

The nonparametric curve estimation using Nadaraya Watson and local linear methods are shown in Figure 1.C. As it is essential to be consistent with the model assumptions stated in previous sections we pre-processed market excess rate to be within (0, 1).

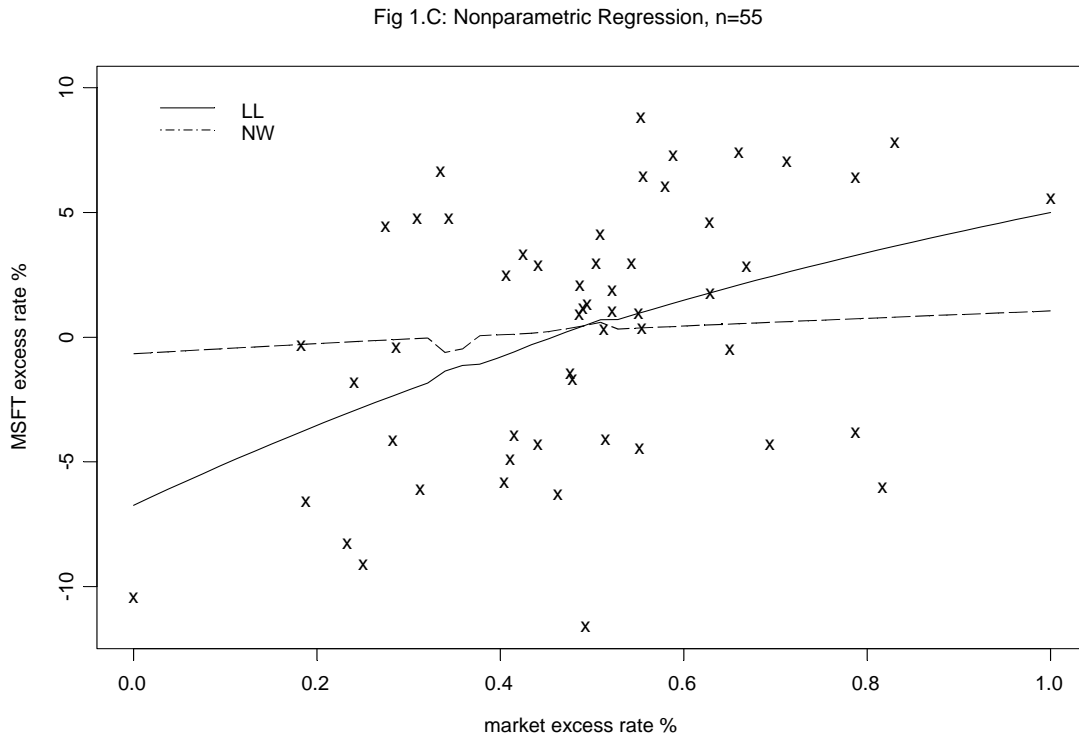


Figure 1.C: Nonparametric regression based on the 55 months of Microsoft–SP500 monthly data.

Next we examine the sequential nonparametric approach. As in the case of our simulation study we first take initial observations of size $n_0=30$, $\alpha=0.05$ and we take $d = 0.75, 1.0$. Next we determined N , the final sample size.

Fig 2.A: Sequential Nonparametric Regression, d=1.0, N=59

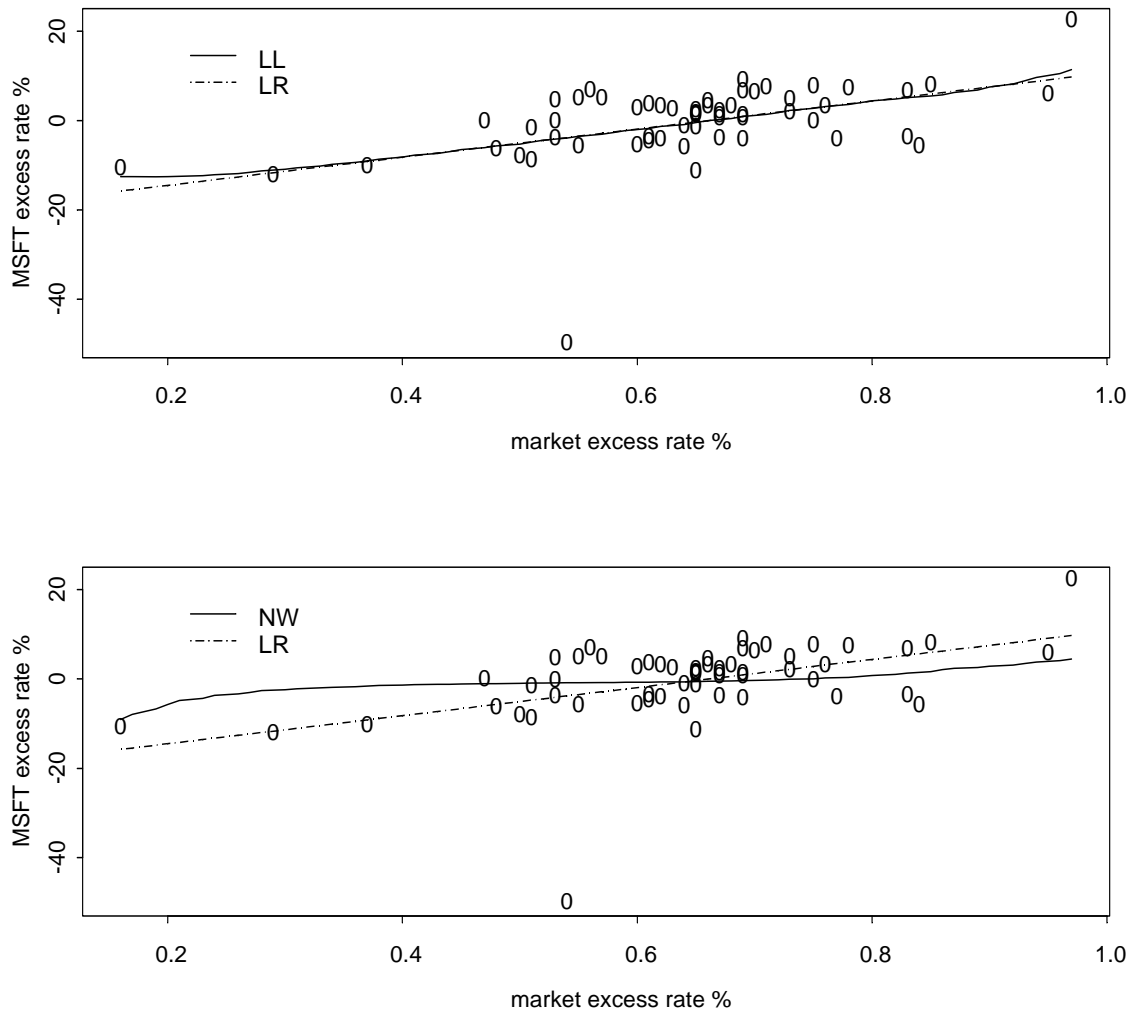


Figure 2.A: Sequential Nonparametric estimates using local linear and Nadaraya-Watson estimates for the data of Microsoft stock. The solid lines are either local linear (LL) or Nadaraya-Watson (NW) estimates and dotted lines are linear regression estimates.

$$\hat{\beta}_0 = -1.3480 \text{ and } \hat{\beta}_1 = 1.2917.$$

Fig 2.B: Sequential Nonparametric Regression, $d=0.75$, $N=120$

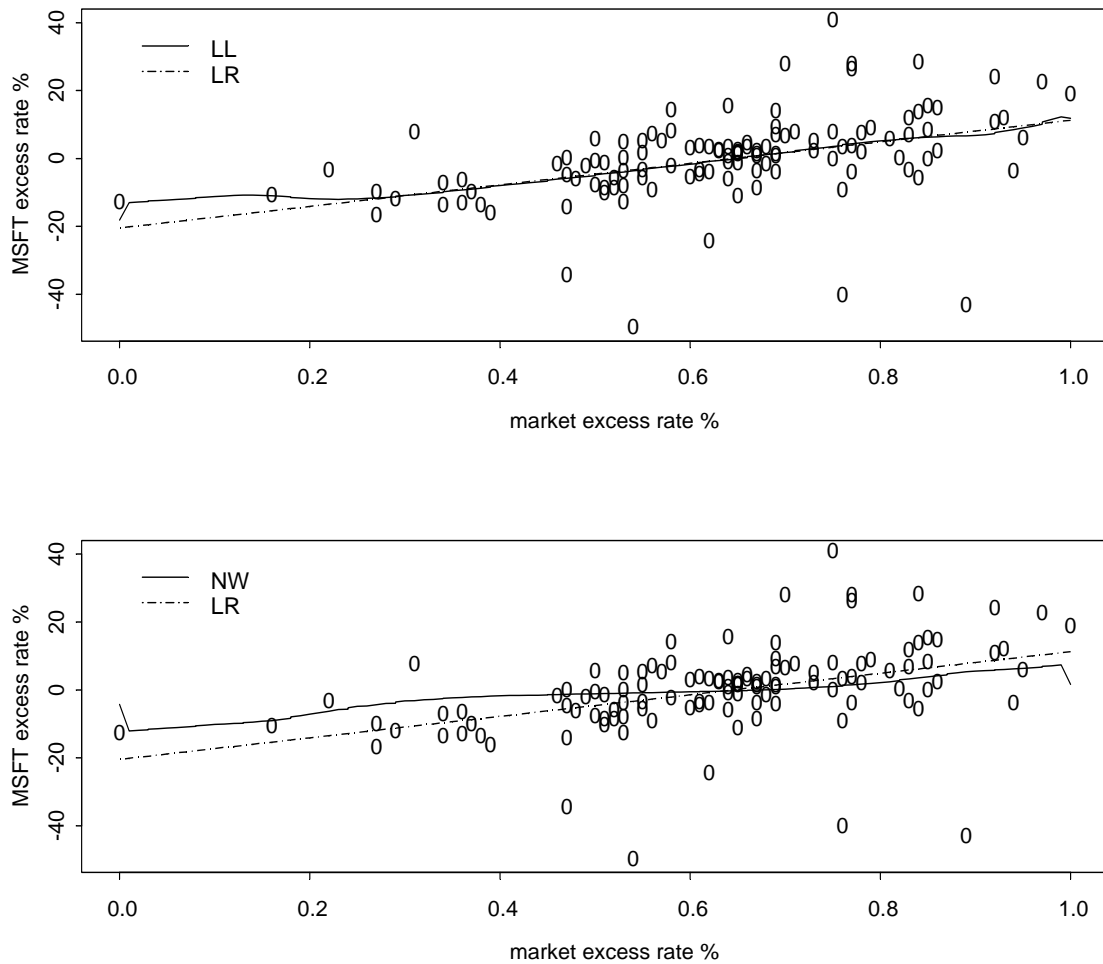


Figure 2.B: Sequential Nonparametric estimates using local linear and Nadaraya-Watson estimates for the data of Microsoft stock. $\hat{\beta}_0 = -0.8686$ and $\hat{\beta}_1 = 1.3057$.

Nadaraya Watson curve estimation shows larger absolute values of the market returns imply smaller absolute values of the asset returns compared to those predicted by linear model. Local linear curve estimation shows smaller values of the market returns imply smaller values of the asset returns and vice a versa.

We may conclude that sequential nonparametric regression can be used in assessing return on risky assets. Rather than relying on beta's disclosed by companies this could be used as an additional analytical tool for investors to know more about assets in their portfolio.

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